Int. J. Heat Mass Transfer. Vol. 31, No. 3, pp. 505–515, 1988
Printed in Great Britain

0017–9310/88 $3.00+0.00
Pergamon Press plc

Heat conduction numbering system for basic geometries

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(*Received* 2 *December* 1986 *and in final form* 11 *May* 1987)

Abstract—A numbering system for transient heat conduction solutions is proposed. It builds on previous usage of the descriptions of boundary conditions. One-, two- and three-dimensional geometries are included. Furthermore, a unique numbering system is proposed to describe boundary, interface, and initial conditions. Examples of the use of the notation are given. Advantages of the system are noted.

1. INTRODUCTION

THE NUMBER of exact solutions in transient heat conduction and diffusion is extremely large and is growing. These solutions are needed for thermal modeling of various devices, as test cases for finite difference/element programs, and as influence functions for the unsteady surface element method [1, 2]. Solutions are given in many different pamphlets, and government and industry reports. Due to the lack of organization of the solutions, it was formerly easier to rederive a solution than to search for it. With the advent of extremely large and relatively inexpensive computer memories, the development of specialized data bases has become practical and they exist in medicine, law and many other fields. One paper on using artificial intelligence for heat transfer problems has been written by Sharma and Minkowycz [3]. Data bases are used by expert systems. In developing such systems, it is very useful to have a numbering system to organize the information.

The purpose of this paper is to propose a numbering system for transient heat conduction. Such a system will not only simplify construction of a computer data base but it will make deriving new solutions simpler and locating solutions less tedious.

This paper deals with exact solutions. However, the numbering system can be employed for nonlinearities caused by temperature-variable properties. Basic geometries such as plates, cylinders and spheres are considered. Irregular geometries such as plates with several randomly spaced holes are not covered.

A numbering system for the basic geometries and boundary conditions is given in refs. [4, 5] and is described in the next section. That numbering system is used as a starting point in this paper. The proposed numbering system extends the notation to describe the initial spatial temperature variation and the time-variation of the boundary conditions. In addition, interface conditions between bodies are covered.

The numbering system in this paper is specifically developed for transient diffusion and heat conduction. The same concepts, however, are applicable to other fields such as convective heat transfer, fluid mechanics and wave phenomena. Steady state is also covered because it is included by the transient notation.

The plan of this paper is to first review the numbering system for geometry and boundary conditions in Section 2. A new scheme for describing interface conditions is given in Section 3. Section 4 provides a numbering system that describes the time and/or space variations of the non-homogeneous term at a boundary. Section 5 gives an initial temperature distribution numbering system and Section 6 gives some examples of the numbering system. Section 7 discusses some advantages of the numbering system; these include aiding in the development and use of a data base in transient conduction, development of an algebra for linear cases, and aiding in derivation of exact solutions.

2. GEOMETRY AND BOUNDARY CONDITION NUMBERING SYSTEM

A numbering system for basic geometries and types of boundary conditions is given in refs. [4, 5]. For the rectangular coordinate system the symbol X is used to denote the x-coordinate, Y is used to denote the y-direction and Z is used to denote the z-direction. For a two-dimensional problem, X and Y are used; for a three-dimensional problem, X, Y and Z are used. The

NOMENCLATURE

b	film thickness	RS	notation for spherical radial coordinate
B	notation for boundary condition modifier	t	time
c	specific heat	$T(\mathbf{r}, t)$	temperature
C	notation for interface condition	T	notation for initial condition
$f(t)$	non-homogeneous boundary condition term	x	space coordinate
$F(x)$	initial temperature distribution	X	notation for rectangular coordinate in x-direction
$G(x, t\|x', \tau)$	Green's function	y	space coordinate
h	heat transfer coefficient	z	space coordinate.
k	thermal conductivity		
n	normal coordinate	Greek symbols	
r	radial coordinate	θ	spherical angular coordinate
R	notation for cylindrical radial coordinate	ρ	density
		ϕ	angular coordinate.

three-dimensional equation for transient conduction with constant, isotropic thermal conductivity, k, is

$$k\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right]=\rho c\frac{\partial T}{\partial t} \tag{1}$$

where T is temperature, ρ is density, c is specific heat and t is time. Though k and ρc are considered constant in most analytical solutions, a method for including a reference to variable properties is given in the proposed numbering system.

For the cylindrical coordinates, r, ϕ, x, the symbol R is for r, Φ is for the angle ϕ and X is for the axial coordinate. For constant k, the three-dimensional equation is

$$k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}+\frac{\partial^2 T}{\partial x^2}\right]=\rho c\frac{\partial T}{\partial t}. \tag{2}$$

For the spherical coordinate system, r, ϕ, θ, the symbols are RS, Φ, Θ. The symbol RS is used to denote radial in the spherical direction. The angle ϕ for both the cylindrical and spherical coordinate goes from 0 to 2π.

Six different boundary conditions are given in refs. [4, 5] and are numbered 0, 1, 2, 3, 4 and 5 (see Table 1). The first, second, and third kinds are commonly denoted as such in the American and Russian literature. In Ozisik's recent excellent book [6] on heat conduction, these three boundary conditions are also referred to as the first, second and third kinds. They are also used in the book by Mikhailov and Ozisik [7]. Carslaw and Jaeger [8] discuss all the boundary conditions mentioned above, including the fourth and fifth kinds, but do not use the words first kind, etc. In the Russian translation of the book by Luikov [9], the words "boundary conditions of the first, second and third kind" are consistently used. Unfortunately what Luikov calls the "boundary condition of the fourth kind" is not really a boundary condition but a variety of composite body problems.

The *first* kind of boundary condition (Dirichlet) is the prescribed temperature at boundary i

$$T(\mathbf{r}_i, t) = f_i(\mathbf{r}_i, t) \tag{3}$$

where $f_i(\mathbf{r}_i, t)$ is the space and time dependent surface temperature. For a one-dimensional case at $x = 0$, $f_i(\cdot)$ can be a function of time only, such as $T(0, t) = f_1(t)$. For a two-dimensional case with coordinates, (x, y), and $x = x_1$

$$T(x_1, y, t) = f_1(y, t).$$

The *second* kind of boundary condition (Neumann) is prescribed heat flux

$$k\left.\frac{\partial T}{\partial n_i}\right|_{\mathbf{r}_i} = f_i(\mathbf{r}_i, t) \tag{4}$$

where n_i is an outward pointing normal. For a one-dimensional case of boundaries at $x_1 = 0$ and $x_2 = L$, $n_1 = -x$ and $n_2 = x$; the boundary conditions are

Table 1. Types of boundary conditions

Notation	Name of boundary condition	Description of boundary condition
0	Zeroth kind (natural)	No physical boundary
1	Dirichlet	Prescribed temperature, equation (3)
2	Neumann	Prescribed heat flux, equation (4)
3	Robin	Convective condition, equation (6)
4	Fourth kind (Carslaw)	Thin film, no convection, equation (7)
5	Fifth kind (Jaeger)	Thin film, convection, equation (8)

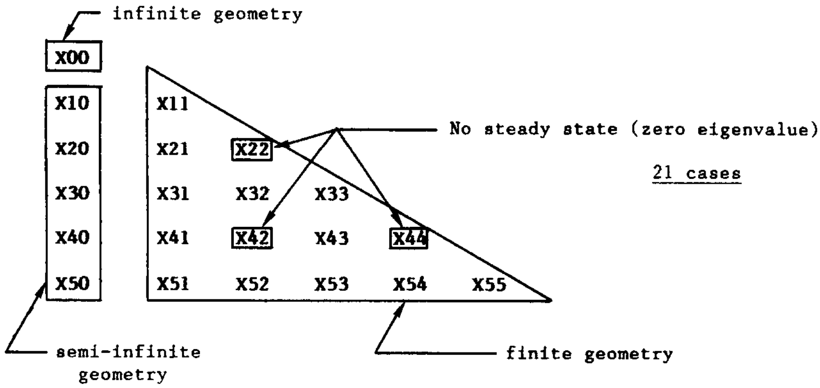

FIG. 1. Distinct cases for one-dimensional Cartesian geometries.

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = f_1(t), \quad k\frac{\partial T}{\partial x}\bigg|_{x=L} = f_2(t) \quad \text{(5a, b)}$$

and $f_1(t)$ and $f_2(t)$ are heat fluxes directed toward the surfaces.

The *third* kind (Robin) is a convective condition

$$k\frac{\partial T}{\partial n_i}\bigg|_{\mathbf{r}_i} + h_i T|_{\mathbf{r}_i} = f_i(\mathbf{r}_i, t) \tag{6}$$

where h_i is the heat transfer coefficient and $f_i(\mathbf{r}_i, t)$ is usually equal to $h_i T_\infty$ with T_∞ being the ambient temperature, but $f_i(\mathbf{r}_i, t)$ can also include a prescribed heat flux.

The *fourth* kind (Carslaw) is for a thin film at a surface with a prescribed heat flux, $f_i(\cdot)$

$$k\frac{\partial T}{\partial n_i}\bigg|_{\mathbf{r}_i} = f_i(\mathbf{r}_i, t) - (\rho cb)_i \frac{\partial T}{\partial t}\bigg|_{\mathbf{r}_i}. \tag{7}$$

The product $(\rho cb)_i$ is for the film at the ith surface and b_i is its thickness. This boundary condition can also describe a well-stirred fluid.

The *fifth* kind (Jaeger) of boundary condition is also for a thin film but permits heat losses from the film by convection

$$k\frac{\partial T}{\partial n_i}\bigg|_{\mathbf{r}_i} + h_i T = f_i(\mathbf{r}_i, t) - (\rho cb)_i \frac{\partial T}{\partial t}\bigg|_{\mathbf{r}_i}. \tag{8}$$

Another important case is the *zeroth* kind. It is for conditions for which there is no physical boundary; it is sometimes called a natural boundary condition. It includes several cases. In the rectangular coordinates, the zeroth kind exists when a boundary extends to infinity. For example, a semi-infinite body that is convectively heated at $x = 0$ is denoted $X30$. Another is for the center of radial cylindrical and spherical bodies that are solid. A solid cylinder with a prescribed surface heat flux is denoted $R02$. The case associated with a convective boundary condition at $r = a$ and a spherical domain outside $r = a$ is denoted $RS30$. Another case is for a thin annular ring which is denoted $\Phi00$.

Cases included in this numbering system are illustrated by Figs. 1–3. The first is for the Cartesian coordinate x and includes 21 distinct cases; others such as $X12$ can be listed but these can be found by a simple change of coordinates (i.e. $x \to L-x$, where L is the plate thickness). Notice that the cylindrical radial case shown in Fig. 2 includes 26 cases because the $RI0$ $(I = 1, \ldots, 5)$ geometries are quite different from the $R0I$ geometries, the former being the infinite region bounded internally by the radius $r = a$ and the latter for solid cylinders of radius a. The annular geometries have neither I nor J in RIJ equal to zero and have boundary radii of a and b. The spherical radial cases,

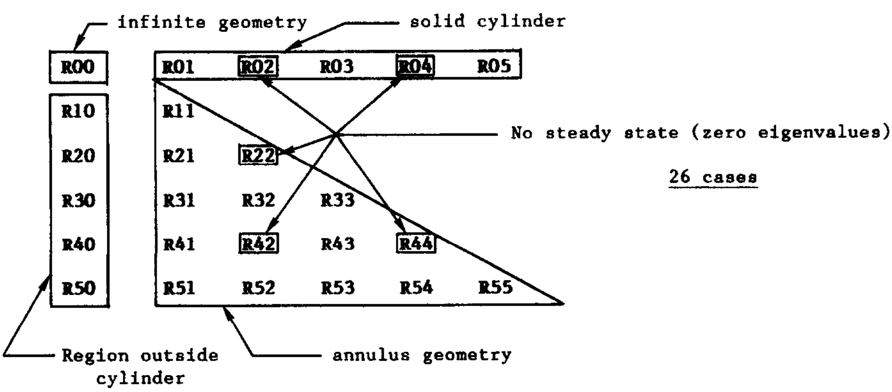

FIG. 2. Distinct cases for one-dimensional radial geometries.

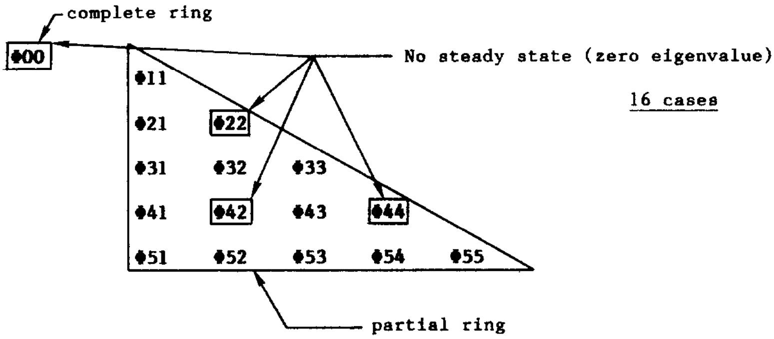

FIG. 3. Distinct cases for ring geometries

RSIJ, can be displayed as Fig. 2 with *R* replaced by *RS*. For the cylindrical coordinate ϕ and small changes in *r*, a ring is obtained; cases are displayed in Fig. 3. The special case in Fig. 3 is for a complete ring. There are neither $\Phi 0I$ nor $\Phi I0$ cases with $I \neq 0$. Except for the $\Phi 00$ case, the cases in Fig. 3 have the same mathematical solution as the corresponding *XIJ* cases of Fig. 1.

There are three special *finite*-body cases in Fig. 1 which (usually) have no steady state; namely, *X*22, *X*42 and *X*44. There are five such special cases in Fig. 2 and four in Fig. 3. Mathematically, these cases are associated with zero eigenvalues. From a physical perspective, these cases do not have a steady state for time-independent values of $f_i(\cdot)$ in equation (4) or (7) (unless there is the special case of zero net heat added). The $\Phi 00$ case is unique since there are no physical boundaries; however, in this case (and the special finite body cases) there is no steady state for a constant volume source in the respective bodies.

For the *infinite* geometries of Figs. 1 and 2, i.e. the first column in both figures, steady state is not usually attained in finite times.

3. INTERFACE DESCRIPTORS

The proposed numbering system includes composite bodies. The interface conditions are denoted in a manner similar to the boundary conditions. For perfect contact, the letter *C* is used for each side of the interface.

For other conditions the letter *C* is followed by a single digit and enclosed in parentheses (see Table 2). The notation, (*C*2), is used to denote a perfect contact with a heat source at the interface. This is a gradient condition and hence is analogous to the second kind of boundary condition. The notation, (*C*3), is used to denote an imperfect contact that can be described by a contact conductance, h_c

$$-k^- \left.\frac{\partial T}{\partial r}\right|_{r_i^-} = h_c(T_{r_i^-} - T_{r_i^+}) = -k^+ \left.\frac{\partial T}{\partial r}\right|_{r_i^+}. \quad (9)$$

The (*C*4) case is for a thin film (or well-stirred fluid) in perfect contact at the interface

$$-k^- \left.\frac{\partial T}{\partial r}\right|_{r_i^-} = (\rho cb)_i \left.\frac{\partial T}{\partial t}\right|_{r_i^+} - k^+ \left.\frac{\partial T}{\partial r}\right|_{r_i^+} \quad (10)$$

where $(\rho cb)_i$ is for the thin film or well-stirred fluid. The (*C*5) interface condition is

$$-k^- \left.\frac{\partial T}{\partial r}\right|_{r_i^-} = h_c^-(T_{r_i^-} - T_{r_i}) + (\rho cb)_i \left.\frac{\partial T}{\partial t}\right|_{r_i}$$

$$+ h_c^+(T_{r_i} - T_{r_i^+}) = -k^+ \left.\frac{\partial T}{\partial r}\right|_{r_i^+}. \quad (11)$$

There is a thin film at the interface with contact conductances on both sides (or a well-stirred fluid with heat transfer coefficients on both sides). The subscript r_i is for the thin film; r_i^- is for the interface on the left; and r_i^+ is for the interface on the right-hand side.

Table 2. Types of interface conditions

Notation	Description of interface condition
C	Perfect contact
(*C*2)	Perfect contact with source at interface
(*C*3)	Finite contact conductance
(*C*4)	Thin film at interface, perfect contact
(*C*5)	Thin film, finite contact conductances

4. BOUNDARY CONDITION MODIFIERS

The boundary conditions of the first through fifth kinds are denoted as indicated in Section 2 but the time and/or space variation must also be specified. This means that the function $f_i(\mathbf{r}_i, t)$ in equations (3), (4), and (6)–(8) must be described. For one-dimensional cases, f_i is only a function of time. The one-dimensional case is first considered and then two- and three-dimensional cases are discussed.

For one-dimensional cases the function $f_i(t)$ includes zero (denoted *B*0), constant with time (*B*1) (actually a step increase at $t = 0$), linear with time (*B*2), some power other than 1 of *t* (*B*3), exponentials (*B*4), two or more step changes (*B*5), and sinusoids (*B*6) (see Table 3). Only the basic cases are given specific notation. Solutions permitting an arbitrary

Table 3. Types of time- and space-variable function at boundary conditions

Notation	Time-variable boundary function	Notation	Space-variable boundary function (two-dimensional)
$B-$	Arbitrary $f(t)$	$Bx-$	Arbitrary $f(x)$
$B0$	$f(t) = 0$		
$B1$	$f(t) = C$		
$B2$	$f(t) = Ct$	$Bx2$	$f(x) = Cx$
$B3$	$f(t) = Ct^p$	$Bx3$	$f(x) = Cx^p$
$B4$	$f(t) = \exp(-at)$	$Bx4$	$f(x) = \exp(-ax)$
$B5$	step changes in $f(t)$	$Bx5$	step changes in $f(x)$
$B6$	$\sin(\omega t+E)$, $\cos(\omega t+E)$	$Bx6$	$\sin(\omega x+E)$, $\cos(\omega x+E)$

time variation are indicated by a dash, – (see Table 5).

For homogeneous one-, two-, or three-dimensional bodies, the geometry and boundary condition descriptors are followed by the boundary condition modifier, BIJ. An example is $X12B14$ where $B14$ indicates that the boundary condition of the first kind (prescribed temperature) at $x = 0$ is a non-zero constant and the boundary condition of the second kind (prescribed q) at $x = L$ has an exponential dependence on time. In general, two indices follow B but there is an exception. Only one index is needed when there is a boundary condition of the zeroth kind such as $X20B1$ or $R03B1$, where the $B1$'s describe the non-zero boundary conditions. If both boundaries are of the zeroth kind (e.g. $X00$, $R00$ and $\Phi00$), then the B modifier is not used.

For two-dimensional cases the variation of $f(\cdot)$ at a boundary can be a function of space as well as time. For a two-dimensional problem involving x and y coordinates and at a y-surface, $f(\cdot)$ could be a function of x alone, a function of t alone or a function of x and t. If $f = f(x)$, then the boundary condition is denoted BxI, $I = 2,\ldots,6$ (since $I = 0$ and 1 are not needed here). If $f = f(x, t)$, then the notation $B(xItJ)$ (where I is for x and J for t) can be used. Generalization to three-dimensional cases is direct; e.g. $f = f(x, z, t)$ has the modifier $B(xIzJtK)$ with appropriate values of I, J and K corresponding to x, z and t. The parentheses are used to enclose notation for a single boundary.

5. INITIAL TEMPERATURE DISTRIBUTION

The initial temperature distribution is given in general coordinates by

$$T(\mathbf{r},0) = F(\mathbf{r}) \tag{12}$$

and for a one-dimensional case with x being the coordinate

$$T(x,0) = F(x). \tag{13}$$

A numbering system for $F(\cdot)$ is proposed that is analogous to that for the boundary conditions (see Table 4). The coordinate r in Table 4 can represent any single space coordinate such as r, x or ϕ. Figure 4 displays some one-dimensional cases and gives the numbers including the notation for the initial temperature distribution. For two- and three-dimensional cases see Figs. 5 and 6 which are discussed in Section 6.

Table 4. Types of space-variable initial conditions

Notation	Single space-variable initial condition
$T-$	Arbitrary $F(r)$
$T0$	$F(r) = 0$
$T1$	$F(r) = C$
$T2$	$F(r) = Cr$
$T3$	$F(r) = Cr^p$
$T4$	$F(r) = \exp(-ar)$
$T5$	step changes in $F(r)$
$T6$	$\sin(\omega r+E)$, $\cos(\omega r+E)$

6. EXAMPLES OF NUMBERING SYSTEM

The proposed numbering system can be used to describe tens of thousands of one-dimensional cases and more than millions for two- and three-dimensional cases. Some one-dimensional cases are shown in Fig. 4.

The first four cases of Fig. 4 are for the same basic case of $X21$. Figure 4(a) depicts a plate with a constant heat flux at $x = 0$ (boundary condition of the second kind) and $T = 0$ at $x = L$ (condition of the first kind). The initial temperature is zero. The number for this is $X21B10T0$ where the 1 following B is for $q = C$ at $x = 0$ and the 0 following $B1$ is for the $T = 0$ condition at $x = L$ (see Table 3). The problem of Fig. 4(b) is insulated at $x = 0$, has a linear time variation of temperature at $x = L$ and has a zero initial temperature; its number is $X21B02T0$. The two in $B02$ is for the linear time variation at $x = L$. Figure 4(c) has $f = 0$ at both boundaries but the initial temperature is a linear function of x and thus is denoted $X21B00T2$. The case shown by Fig. 4(d) includes all the non-zero f_i and F values of Figs. 4(a)–(c).

The composite plate case shown in Fig. 4(e) has two plates that are in perfect contact. At $x = 0$ there is a constant heat flux and at the other boundary there is a convective condition with the ambient temperature varying in a sinusoidal manner. The initial temperature in the first plate is zero and 5°C in the second plate. The number for this case is given on the figure.

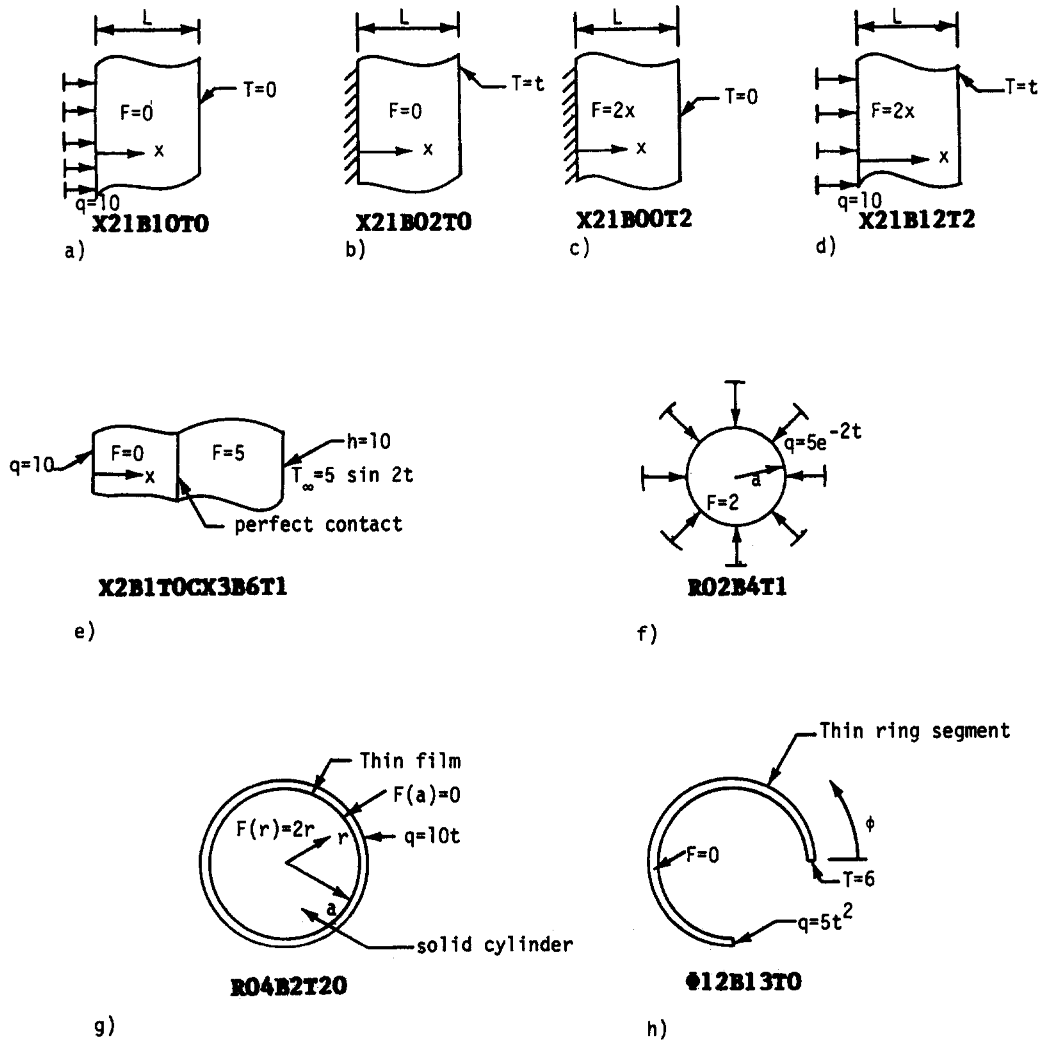

FIG. 4. Some one-dimensional examples of the numbering system.

Two cylindrical radial cases are shown in Figs. 4(f) and (g). Figure 4(f) depicts a solid cylinder with a heat flux of exponential form at $r = a$ and the initial temperature is a constant. Figure 4(g) is for a thin film at the surface of a solid cylinder. There is an applied heat flux at the surface of the thin film (or there is an equivalent volume heating in the film). The initial temperature in $0 < r < a$ is $F(r) = 2r$ while the initial temperature of the film is zero. For such cases the T symbol must be followed by the indices for the interior of the body and the film. They are given in the order which they appear, starting $r = 0$. For this case the number is $R04B2T20$. Figure 4(h) is for a segment of a thin ring.

Some two-dimensional cases are illustrated in Fig. 5. A rectangular plate is shown in Fig. 5(a). The number description in the x-direction is similar to that for a one-dimensional case and it is then followed by the one in the y-direction. Since the initial temperature is known to be zero, it is redundant to repeat this information with the y-direction notation. Another two-dimensional case is shown by Fig. 5(b); it is for a plate that is finite in the x-direction and semi-infinite in the y-direction. For the x-direction the boundary conditions are of the second and first kinds and are homogeneous but the initial temperature distribution is linear with x; thus this part of the notation is $X21B00T2$. For the y-direction there is a step increase in q at $x = 0$ and a step decrease at $x = b$ and there is no physical boundary for large y. Hence, the notation in the y-direction is $Y20Bx5$ where the $Bx5$ notation is for the steps in q in the x-direction at the $y = 0$ boundary. There is no y-direction dependence of the initial temperature so it is omitted in the notation.

A case of a body outside the cylindrical radius of $r = a$ is shown by Fig. 5(c). There is a sinusoidal variation with ϕ of the surface heat flux and the initial temperature distribution is constant. The notation is

Table 5. Some basic one-dimensional cases in Carslaw and Jaeger [8]

Number	Ref.	Page	Equation	Comments
$X00T5$	CJ	54	3	$T(x,0) = T_0 - a < x < a$; $T(x,0) = 0$, $\lvert x\rvert > a$
$X10B1T0$	CJ	60	10	
$X10B3T0$	CJ	305	6	$T(0,t) = T_0 t^{n/2}$, $n = 1,2,\ldots$
$X11B00T1$	CJ	96	6	
$X12B01T0$	CJ	113	6	Better for small dimensionless times
$X12B01T0$	CJ	113	5	Better for large dimensionless times
$X13B10T0$	CJ	126	16	
$X13B01T0$	CJ	125	15	
$X20B1T0$	CJ	75	6,7	The forms of solution
$X21B01T0$	CJ	100	2,4	
$X22B01T0$	CJ	112	4	Better for small dimensionless times
$X22B01T0$	CJ	112	3	Better for large dimensionless times
$X23B00T1$	CJ	122	12	$T(x,0) = T_0$, $0 < x < L$
$X23B10T0$	CJ	125	14	
$X23B02T0$	CJ	127	9	$T_\infty = kt$
$X24B01T00$	CJ	128	5	
$X30B1T0$	CJ	306		Second equation on page
$X33B00T-$	CJ	126	21	Arbitrary initial temperature
$R01B0T1$	CJ	199	5	
$R01B1T0$	CJ	331	3	Small time solution
$R02B1T0$	CJ	203	1	
$R03B1T0$	CJ	202	8	
$R11B00T1$	CJ	207	13	
$R21B10T0$	CJ	334	12	
$RS01B0T1$	CJ	348	6	
$RS02B1T0$	CJ	242	1	
$RS03B0T1$	CJ	238	10	

$R20B\phi6T1\Phi00$. The $B\phi6$ describes the boundary condition at $r = a$ and no index is needed for $r \to \infty$ since there is no physical boundary.

Figure 5(d) displays a semi-infinite cylinder that is insulated at all surfaces except at the center at the top where a circular heat flux is applied. The initial temperature is zero. The number for this case is $R02B0T0X20Br5$ where the $Br5$ notation is used because the heat flux is not constant with r but can be considered to have a step increase at $r = 0$ and a step decrease at $r = a$. If the heat flux were over the circular region shown and also varied as ct in time, $Br5$ would be replaced by $B(r5t2)$ where the parentheses are used to denote that both conditions apply at the same boundary.

The numbering system readily extends to three-dimensional cases such as given in Fig. 6. The first case is for a semi-infinite rod that is insulated on all surfaces except there is a constant heat flux over a rectangular region at $z = 0$. The case of a rectangular block is shown in Fig. 6(b), where front and side views are shown.

7. ADVANTAGES OF NUMBERING SYSTEM

There are several types of advantages of the proposed numbering system. The first to be discussed in a subsection relates to a data base of conduction solutions. The second relates to an algebra that can be given for linear problems. The last major advantage to be discussed relates to use of the method in conjunction with Green's functions to obtain solutions for linear problems.

7.1. *Data base in transient heat conduction*

One of the obvious advantages of a numbering system is that it facilitates the organizing of a data base. A structure is provided that makes the storage of solutions easier. Also important is that it greatly reduces the effort in locating solutions. Instead of relying on imprecise verbal titles of papers (or abstracts) to describe a particular problem, a search based on the notation given herein can be much more direct and less prone to overlook related solutions.

The proposed numbering system has been utilized to catalog most of the solutions of Carslaw and Jaeger [8], Luikov [9], Ozisik [6] and other books. It has been found to work very well. An example of a portion of a data base for some basic solutions is given in Table 5. A more extensive data base is available from the author [10]. These solutions are for constant properties but the numbering system can also be used for temperature-variable properties; in such cases, appropriate comments would be added.

Table 5 gives some numbers of some one-dimensional cases for the x, cylindrical radial, and spherical radial cases. The first column contains the number; the second column gives the reference (which is CJ in Table 5), denoting Carslaw and Jaeger [8]; the third and fourth columns give the page and equation num-

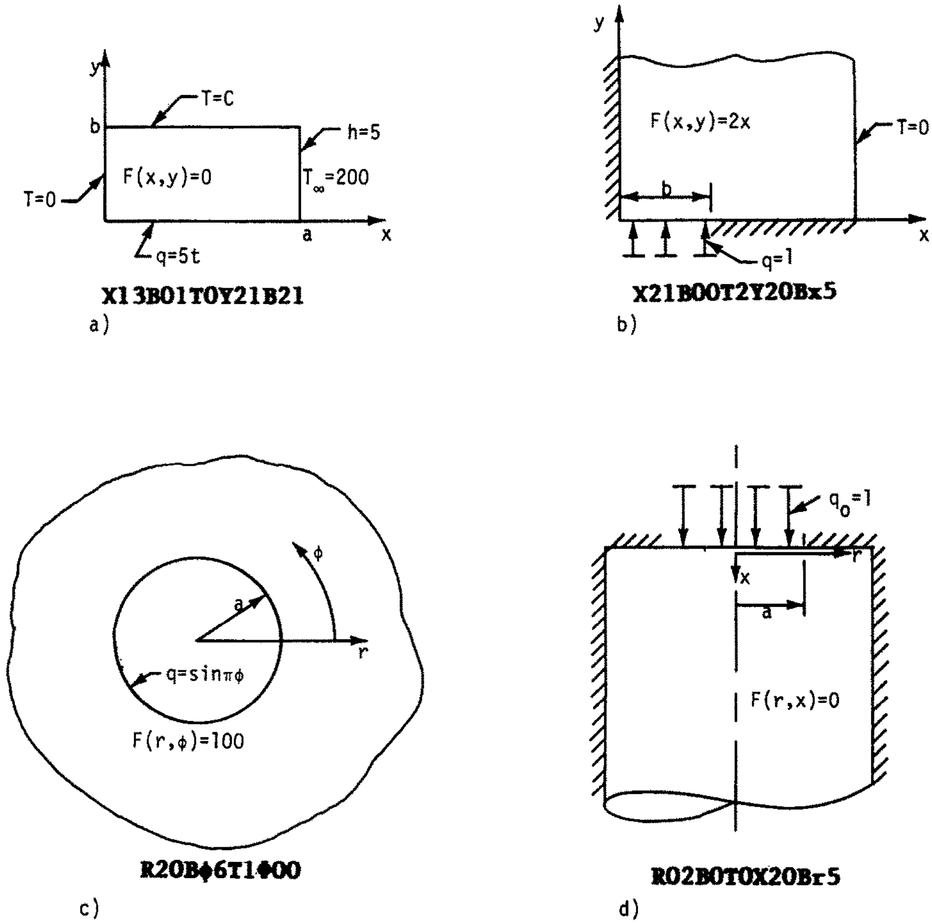

FIG. 5. Two-dimensional examples of the numbering system.

bers of the reference; and the last column contains some comments. In more extensive versions of the computerized data base, the solution could be given, evaluated and plotted.

7.2. *Algebra for linear cases*

For linear cases several kinds of algebraic manipulations are possible. This brief discussion can include only a few possibilities.

One case involves boundary conditions of the zeroth, first, and third kinds and the uniform initial temperature distribution. An example is

$$T_{X10B1T0}|_{T(0,t)=T_0} = T_0(1-T_{X10B0T1}|_{T(x,0)=1}) \quad (14)$$

where T_0 is a constant. The notation means the temperature for the subscripted case.

In addition to relating boundary conditions and the initial temperature, the notation suggests a method of superimposing solutions. The number of non-zero values of the indices following B and T give the number of superposition problems that can be formed; this is the number of 'forcing' terms. An example is provided by the first four cases of Fig. 4. The Fig. 4(d) case is the sum of the first three

$$X21B12T2 = X21B10T0+X21B02T0 + X21B00T2. \quad (15)$$

Notice that $B12$ contains two non-zero digits and $T2$ contains one; hence, the case of Fig. 4(d) can be given as the sum of three problems. The same superposition principles can be used for the two-dimensional problem of Fig. 5(a).

Another type of superposition is possible for more than one forcing term at a boundary. An example is for the Fig. 4(a) case with

$$q_0 = 10+5t. \quad (16)$$

The temperature solution can be written as

$$T|_{q_0=10+5t} = 10T_{X21B10T0}|_{q_0=1} + 5T_{X21B20T0}|_{q_0=t}. \quad (17)$$

7.3. *Use of numbering system for obtaining solutions*

Another important advantage of the numbering system is for aiding in the generation of solutions through the use of Green's functions.

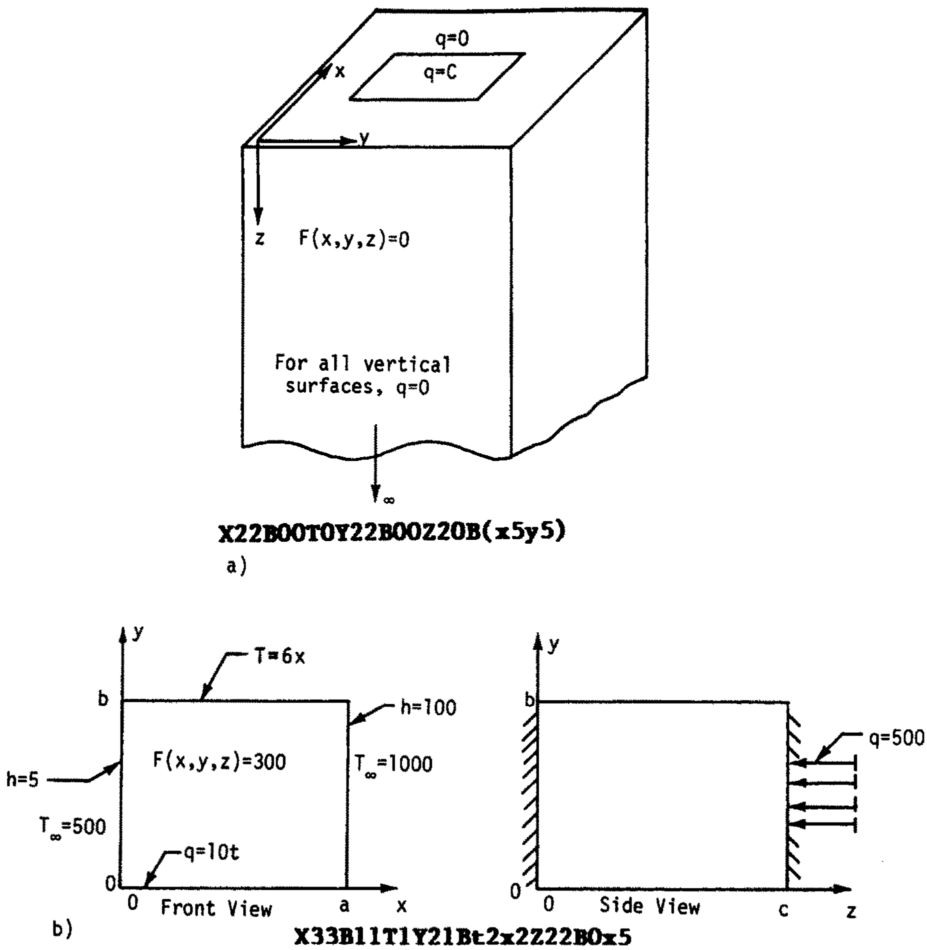

FIG. 6. Three-dimensional examples of the numbering system.

A general form of the Green's function solution for linear transient heat conduction problems is given in ref. [4]. For two-dimensional Cartesian problems such as shown in Figs. 5(a) and (b), the Green's function solution is

$$T(x,y,t)=\int_{x'}\int_{y'}G(x,y,t|x',y',0)F(x',y')\,dx'\,dy'$$

$$+\frac{\alpha}{k}\int_{\tau=0}^{t}\sum_{i=1}^{I}\int_{S_i}f_i(x_j',y_j',\tau)G(x,y,t|x_i',y_i',\tau)\,ds_i'\,d\tau$$

boundary conditions of the second and third kinds

$$-\alpha\int_{\tau=0}^{t}\sum_{j=1}^{J}\int_{S_i}f_i(x_j',y_j',\tau)\left.\frac{\partial G}{\partial n_j'}\right|_{r'=r_j}ds_j'\,d\tau \quad (18)$$

boundary conditions of the first kind only

where $G(x,y,t|x',y',\tau)$ is the Green's function; α is thermal diffusivity; f_i and f_j are the non-homogeneous terms for the i and j boundaries; I is the number of boundary conditions of the second and third kinds; and J is the number of boundary conditions of the first kind. The number of terms in equation (18)

$$I+J+1 \text{ (if } F(x,y)\neq 0) \quad \text{and} \quad I+J \text{ (if } F(x,y)=0)$$

is the same number of non-zero terms following B and T in the notation. For Fig. 5(a), for example, $I=2$ which includes one non-zero boundary condition of the second kind at $y=0$ and one non-zero boundary condition of the third kind at $x=a$; and there is one non-zero boundary condition of the first kind at $y=b$ and thus $J=1$. Since $F(x,y)=0$, there is a total of three parts to the Green's function solution. The solution for Fig. 5(a) can be written as

$$T(x,y,t)=\frac{\alpha}{k}\int_{\tau=0}^{t}\int_{x'=0}^{a}5\tau G(x,y,t|x',0,\tau)\,dx'\,d\tau$$

*X*13*B*00*T*0*Y*21*B*20

$$+\frac{\alpha}{k}\int_{\tau=0}^{t}\int_{y'=0}^{b}1000G(x,y,t|a,y',\tau)\,dy'\,d\tau$$

*X*13*B*01*T*0*Y*21*B*00

$$-\alpha\int_{\tau=0}^{t}\int_{x'=0}^{a}C\frac{\partial G(x,y,t|x',b,\tau)}{\partial y'}dx'\,d\tau \quad (19)$$

*X*13*B*00*T*0*Y*21*B*01

The associated problems are indicated by the numbers below each integral. Another significant relation of the numbering system to the Green's function solution is that the numbering system denotes which Green's function is needed. That is, each $G(\cdot)$ in equation (19) is the same and can be denoted

$$G(\cdot) = G_{X13Y21}(x, y, t|x', y', \tau).$$

Furthermore, for the rectangular coordinate system, the two- and three-dimensional Green's functions can be formed by products of the one-dimensional Green's functions; for this case the relation is simply

$$G_{X13Y21}(x, y, t|x', y', \tau) = G_{X13}(x, t|x', \tau)G_{Y21}(y, t|y', \tau). \quad (20)$$

These one-dimensional Green's functions are tabulated in refs. [4, 5] and elsewhere. By taking advantage of short and long time expressions of the Green's functions, reduced computation and increased accuracy can sometimes be obtained [11].

The principle of multiplying the one-dimensional Green's function can also be used for the x, y coordinates of Fig. 5(b) and the r, x coordinates of Fig. 5(d) but not for the r, ϕ coordinates of Fig. 5(c).

8. SUMMARY AND CONCLUSIONS

The previously proposed numbering system for transient conduction is extended in this paper to describe space variation of the initial temperature distribution, and time and space variation of the boundary conditions. Various interface conditions are also included. The numbering system is shown to have many advantages which include providing a basis for a data base, providing more insight into the solutions, and providing a simplified solution method in connection with Green's functions.

Though the proposed numbering system covers an enormous number of possible cases, the system can readily be extended to cover other conditions such as volume heat sources, fins, and bulk movement of the body. Moreover, the number system provides a prototype for describing other fields such as heat convection and wave motion.

Acknowledgements—This research was supported in part by the National Science Foundation under Grant No. MEA81-21499.

REFERENCES

1. N. R. Keltner and J. V. Beck, Unsteady surface element method, *J. Heat Transfer* **103**, 759–764 (1981).
2. J. V. Beck and N. R. Keltner, Transient contact of two semi-infinite bodies over a circular area. In *Spacecraft Radiative Transfer and Temperature Control* (Edited by T. E. Horton), Vol. 83, *Progress in Astronautics and Aeronautics*, pp. 61–82 (1982).
3. A. Sharma and W. J. Minkowycz, KNOWTRAN: an artificial intelligence system for solving heat transfer problems, *Int. J. Heat Mass Transfer* **25**, 1279–1289 (1982).
4. J. V. Beck, Green's function solution for transient heat conduction problems, *Int. J. Heat Mass Transfer* **27**, 1235–1244 (1984).
5. J. V. Beck, Green's functions and numbering system for transient heat conduction, AIAA Paper No. AIAA-84-1741 and *AIAA J.* **24**, 327–333 (1986).
6. M. N. Ozisik, *Heat Conduction*. Wiley, New York (1980).
7. M. D. Mikhailov and M. N. Ozisik, *Unified Analysis and Solutions of Heat and Mass Transfer*. Wiley, New York (1984).
8. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, 2nd Edn. Oxford University Press, London (1959).
9. A. V. Luikov, *Analytical Heat Diffusion Theory*. Academic Press, New York (1968).
10. L. S. Tzeng and J. V. Beck, Data base for solutions in transient heat conduction, MSU-ENGR-85-018, Michigan State University, College of Engineering (August 1985).
11. J. V. Beck and N. R. Keltner, Green's function partitioning method applied to foil heat flux gages, ASME Paper No. 85-HT-56 and *J. Heat Transfer* **109**, 274–280 (1987).

SYSTEMES DE DENOMBREMENT DE LA CONDUCTION THERMIQUE POUR DES GEOMETRIES DE BASE

Résumé—Un système de dénombrement pour des solutions de conduction thermique variable est proposé. Il construit sur des usages connus des descriptions des conditions aux limites. On inclut des géométries à une, deux et trois dimensions. Un système unique de dénombrement est proposé pour décrire des conditions aux limites, à l'interface et initiales. Des exemples d'utilisation de cette notation sont donnés. On note les avantages de ce système.

NUMMERNSYSTEM FÜR WÄRMELEITUNG IN GRUNDLEGENDEN GEOMETRIEN

Zusammenfassung—Es wird ein Nummernsystem zur Lösung transienter Wärmeleitvorgänge vorgeschlagen. Dieses baut auf früheren Formulierungen der Randbedingungen auf. Ein-, zwei- und dreidimensionale Geometrien werden berücksichtigt. Desweiteren wird ein Nummernsystem zur Beschreibung von Rand-Schnittstellen und Anfangsbedingungen vorgestellt. Die Anwendung der Notation wird an Beispielen gezeigt. Die Vorteile des Systems werden vorgestellt.

СИСТЕМА ЗАПИСИ ЗАДАЧИ ТЕПЛОПРОВОДНОСТИ ДЛЯ ОСНОВНЫХ ГЕОМЕТРИЙ

Аннотация—Предложена универсальная система записи решений задачи нестационарной теплопроводности, основанная на использовании известных граничных условий и включающая одно-, двух- и трехмерные геометрии. Кроме того, предложена универсальная система записи начальных и граничных условий, а также условий на межфазных границах. Приводятся примеры использования этой системы записи и отмечаются ее достоинства.